

Speciality of Lie–Jordan Algebras

A. N. Grishkov

Instituto de Matemática e Estatística Universidade de São Paulo São Paulo Brazil

metadata, citation and similar papers at core.ac.uk

Ivan P. Shestakov¹

*Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil;
and Sobolev Institute of Mathematics, Novosibirsk, Russia*

Communicated by Efim Zelmanov

Received October 1, 1999

The class of so-called Lie–Jordan algebras, which have one binary (Lie) operation $[x, y]$ and one ternary (Jordan) operation $\{x, y, z\}$, that satisfy the identities

$$[x, y] = -[y, x],$$

$$\{x, y, z\} = \{z, y, x\},$$

$$[[x, y], z] = \{x, y, z\} - \{y, x, z\},$$

$$\{\{x, y, z\}, t\} = \{\{x, t\}, y, z\} + \{x, \{y, t\}, z\} + \{x, y, \{z, t\}\},$$

$$\{\{x, y, z\}, t, v\} = \{\{x, t, v\}, y, z\} - \{x, \{y, v, t\}, z\} + \{x, y, \{z, t, v\}\}.$$

is introduced. It is proved that any such algebra is *special*, that is, isomorphic to a subalgebra of a Lie–Jordan algebra of the type A^\pm , obtained from an associative algebra A via the operations $[x, y] = xy - yx$, $\{x, y, z\} = xyz + zyx$. As an application, we prove the conjecture about associativity of a certain loop constructed by A. Grishkov. © 2001 Academic Press

1. INTRODUCTION

It is well known that to every associative algebra A one can relate a Lie algebra $A^- = \langle A, +, [\] \rangle$, with the multiplication $[x, y] = xy - yx$, and a Jordan algebra $A^+ = \langle A, +, \circ \rangle$, with the multiplication $x \circ y = \frac{1}{2}(xy +$

¹ Supported by CNPq Grant 300528/99-0.

yx). Thus, we have the functors $(\cdot)^-$ and $(\cdot)^+$ from the category $\underline{\text{Ass}}$ of associative algebras into the categories $\underline{\text{Lie}}$ and $\underline{\text{Jord}}$ of Lie and Jordan algebras, respectively. It is easy to see that both functors have left adjoint functors $U: \underline{\text{Lie}} \rightarrow \underline{\text{Ass}}$ and $S: \underline{\text{Jord}} \rightarrow \underline{\text{Ass}}$; that is, for any $L \in \underline{\text{Lie}}$, $J \in \underline{\text{Jord}}$, $A \in \underline{\text{Ass}}$ there are bijections

$$\text{Hom}_{\underline{\text{Lie}}}(L, A^-) \rightarrow \text{Hom}_{\underline{\text{Ass}}}(U(L), A),$$

$$\text{Hom}_{\underline{\text{Jord}}}(J, A^+) \rightarrow \text{Hom}_{\underline{\text{Ass}}}(S(J), A),$$

which are functorial in the variables L, J, A . The algebras $U(L)$ and $S(J)$ are called *associative universal enveloping algebras* of L and J , respectively. Furthermore, there are morphisms $\alpha_L^-: L \rightarrow U(L)^-$ and $\alpha_J^+: J \rightarrow S(J)^+$ with the universal properties (these morphisms correspond to the identical morphisms of the algebras $A = U(L)$, $A = S(J)$ in the bijections above).

But at this point the Lie and Jordan cases diverge: the mapping α_L^- is always injective when L is a free module over the ring of scalars, due to the celebrated Poincaré–Birkhoff–Witt theorem, while the mapping α_J^+ could have nonzero kernel even when J is a finite dimensional algebra over a field (see [4, 7]). A Jordan algebra J for which the universal mapping α_J^+ is injective is called *special*. The condition of speciality plays an important role in the theory of Jordan algebras; there are many results and open problems concerning the speciality of particular Jordan algebras and classes of algebras (see [4, 7]).

One may consider other functors that conserve the additive structure of an algebra and introduce for them a notion of speciality (see [6]). In this paper we consider the functor $(\cdot)^\pm$ that corresponds to an associative algebra A the algebra A^\pm , which has the same additive structure and two new multiplications: a Lie one $[x, y] = xy - yx$ and a triple Jordan one $\{x, y, z\} = xyz + zyx$. It is easy to check that the operations $[,]$ and $\{ , , \}$ satisfy the identities

$$[x, y] = -[y, x], \quad (1)$$

$$\{x, y, z\} = \{z, y, x\}, \quad (2)$$

$$[[x, y], z] = \{x, y, z\} - \{y, x, z\}, \quad (3)$$

$$[\{x, y, z\}, t] = \{[x, t], y, z\} + \{x, [y, t], z\} + \{x, y, [z, t]\}, \quad (4)$$

$$\{\{x, y, z\}, t, v\} = \{\{x, t, v\}, y, z\} - \{x, \{y, v, t\}, z\} + \{x, y, \{z, t, v\}\}. \quad (5)$$

Observe that the Jacobi identity follows from (2) and (3).

DEFINITION 1. An algebra with two operations $[,]$ and $\{ , , \}$ is called a *Lie–Jordan algebra* (an LJ-algebra for short) if identities (1)–(5) hold.

Note that an LJ-algebra is a Lie algebra relative to the binary operation and a triple Jordan system relative to the ternary operation.

DEFINITION 2. An LJ-algebra L is called *special* if there exists an associative algebra A such that L is isomorphic to a subalgebra of A^\pm .

The main result of the present article is the following.

THEOREM 1. *Every LJ-algebra over a field of characteristic $\neq 2$ is special.*

In the proof we follow the method developed in [5, 6] and construct the universal enveloping algebra $U(L)$ for a Lie-Jordan algebra L as a deformation of its associated graded algebra $\text{gr } U(L)$.

In the last section we apply this result to prove the conjecture from [2] that a certain loop constructed there is a group.

All the algebras considered in the paper are assumed to be over a field F of characteristic $\neq 2$.

2. UNIVERSAL ENVELOPING ALGEBRA AND ASSOCIATED GRADED ALGEBRA

Let L be an LJ-algebra; then an associative algebra $U(L)$ is said to be a *universal enveloping algebra* for L if there exists a homomorphism $\alpha_L: L \rightarrow U(L)^\pm$ such that for any associative algebra A and a homomorphism $\beta: L \rightarrow A^\pm$ there exists a homomorphism π of associative algebras $\pi: U(L) \rightarrow A$ such that $\beta = \alpha_L \circ \pi$. In other words, there is a bijection

$$\text{Hom}_{\text{Lie-Jord}}(L, A^\pm) \rightarrow \text{Hom}_{\text{Ass}}(U(L), A),$$

which is functorial on the variables L and A .

The existence of a universal enveloping algebra $U(L)$ for a given LJ-algebra L is obvious. It is isomorphic to the quotient algebra of the tensor algebra $T(L)$ by the ideal I generated by all the elements

$$a \otimes b - b \otimes a - [a, b], a \otimes b \otimes c + c \otimes b \otimes a - \{a, b, c\}, a, b, c \in L,$$

with the universal homomorphism $\alpha_L: a \rightarrow a + I$. Theorem 1 is equivalent to the fact that $\ker \alpha = 0$.

The subspace $\alpha(L) \subseteq U(L)$ generates $U(L)$, and so it defines in $U(L)$ an ascending filtration $U_1 = \alpha(L) \subseteq U_2 \subseteq \dots$, where $U_i = U_{i-1} + U_1^i$. Observe that for any $a, b, c \in L$ we have

$$a \otimes b \otimes c - 1/2(\{a, b, c\} + [b, c] \otimes a + b \otimes [a, c] + [a, b] \otimes c) \in I,$$

which implies that $U_2 = U_3 = U(L)$. Consider the associated graded algebra $\text{gr } U = (\text{gr } U)_1 \oplus (\text{gr } U)_2$, where $(\text{gr } U)_1 = U_1$, $(\text{gr } U)_2 = U(L)/U_1$; then it is easy to see that it is commutative and nilpotent of degree 3: $(\text{gr } U)^3 = 0$. For more properties of this algebra we need the following

LEMMA 1. *In $U(L)^\pm$ there hold the inclusions*

$$\begin{aligned}[U_i, U_j] &\subseteq U_{i+j-1}, \\ \{U_i, U_j, U_k\} &\subseteq U_{i+j+k-2}.\end{aligned}$$

Proof. Since $U_2 = U(L)$, both inclusions are evident if at least one of the indices is equal to 2. Furthermore, $U_1 = \alpha(L)$, and for any $a, b, c \in L$ we have

$$[\alpha(a), \alpha(b)] = \alpha([a, b]), \quad \{\alpha(a), \alpha(b), \alpha(c)\} = \alpha(\{a, b, c\}),$$

which proves the inclusions for $i = j = k = 1$. ■

Let $\bar{c}_i = c_i + U_{i-1} \in (\text{gr } U)_i$, for some $c_i \in U_i$. It follows from Lemma 1 that the following operations are correctly defined in $\text{gr } U$:

$$\begin{aligned}[\bar{c}_i, \bar{c}_j] &= [c_i, c_j] + U_{i+j-2}, \\ \{\bar{c}_i, \bar{c}_j, \bar{c}_k\} &= \{c_i, c_j, c_k\} + U_{i+j+k-3}.\end{aligned}$$

Since these operations are induced by those in the LJ-algebra $U(L)^\pm$, they satisfy identities (1)–(5); that is, the space $\text{gr } U$ forms an LJ-algebra with respect to the operations $[\ , \], \{ \ , \ , \}$. Moreover, one can easily check that the following identities hold in $U(L)$,

$$\begin{aligned}[xy, z] &= x[y, z] + [x, z]y, \\ \{x \circ y, z, t\} &= \{x, z, t\} \circ y + 1/2([[t, z], y] \circ x + [[t \circ z, y], x]),\end{aligned}$$

where $x \circ y = xy + yx$. The proof is straightforward. As a corollary, we obtain that the associated graded algebra $\text{gr } U$ satisfies the identities

$$[xy, z] = x[y, z] + [x, z]y, \tag{6}$$

$$\{xy, z, t\} = \{x, z, t\}y + 1/2([[t, z], y]x + [[tz, y], x]). \tag{7}$$

Identity (6) shows that $\text{gr } U$ is a *Poisson algebra*. By this reason, we will call a commutative associative algebra A with additional operations $[\ , \]$ and $\{ \ , \ , \}$ a *Poisson LJ-algebra* if $\langle A, +, [\ , \], \{ \ , \ , \} \rangle$ is an LJ-algebra, $A^3 = 0$, and identities (6) and (7) are satisfied in A . As we just have seen, for every LJ-algebra L , the algebra $\text{gr } U(L)$ is a Poisson LJ-algebra.

LEMMA 2. *Every Poisson LJ-algebra A satisfies the identities*

$$\{x, z, t\}y - \{y, z, t\}x - h(x, y, z, t) = 0, \quad (8)$$

$$[xy, zt] = \{xy, z, tw\} = \{xy, zt, w\} = 0, \quad (9)$$

where $h(x, y, z, t) = 1/2([z, [x, y]]t + [t, [x, y]]z + [[t, z], x]y - [[t, z], y]x)$.

Proof. Interchanging x and y in (7) and subtracting the new identity from the former one, by the Jacobi identity and (6), we obtain (8). Furthermore, since $A^3 = 0$, identities (9) follow from (7), (2), (3), and (6). \blacksquare

Notice that $\alpha(L) = U_1 = (\text{gr } U)_1$; hence the problem of injectivity of α is reduced to the structure of the graded algebra $\text{gr } U(L)$. In the next section we will try to construct this algebra starting with the algebra L .

3. THE POISSON LJ-ALGEBRA $\overline{V(L)}$

Let L be an LJ-algebra. In this section we will construct a Poisson LJ-algebra $\overline{V(L)}$ which eventually proves to be isomorphic to the algebra $\text{gr } U(L)$.

As the first approximation, we consider the space $V(L) = L \oplus S^2(L)$. Here $S^2(L)$ stands for the space of symmetric 2-degree tensors over L , which is spanned by the elements $a \bullet b = 1/2(a \otimes b + b \otimes a)$, $a, b \in L$. Define a multiplication \cdot on $V(L)$ by setting for $a, b, c, d \in L$

$$a \cdot b = a \bullet b, \quad a \cdot (b \bullet c) = (b \bullet c) \cdot a = (a \bullet b) \cdot (c \bullet d) = 0;$$

then $V(L)$ becomes a commutative associative algebra with $V(L)^3 = 0$. If no ambiguity occurs, we will write ab instead of $a \cdot b$.

Now extend the operations $[,]$ and $\{ , , \}$ from L to $V(L)$ by setting

$$\begin{aligned} [x \bullet x, y] &= -[y, x \bullet x] = 2x[x, y], \\ \{x \bullet x, y, z\} &= \{z, y, x \bullet x\} \\ &= \{x, y, z\}x + 1/2([[z, y], x]x + [[zy, x], x]), \\ \{x, y \bullet y, z\} &= \{y \bullet y, x, z\} + [[x, y^2], z], \\ \{a, b, r\} &= \{a, r, b\} = \{r, a, b\} = [a, b] = 0, \end{aligned}$$

for every $x, y, z \in L$, $a, b \in S^2(L)$, $r \in V(L)$. Since the space $S^2(L)$ is spanned by the elements $x \bullet x$, $x \in L$, the operations are correctly defined.

Note that L is an LJ-subalgebra of $V(L)$ though $V(L)$ itself, in general, is not an LJ-algebra with respect to the defined operations.

By construction, we have a homomorphism of graded algebras

$$\rho: \langle V(L), +, \cdot, [\ , \], \{ \ , \ , \} \rangle \rightarrow \text{gr } U$$

such that $\rho(a) = \alpha(a)$, $a \in L$.

Obviously, the equality that we need, $\ker \alpha = 0$, is equivalent to the equality $(\ker \rho) \cap L = 0$. Since $\ker \rho$ is a graded ideal, this equality is equivalent to the inclusion

$$\ker \rho \subseteq S^2(L). \quad (10)$$

Therefore, we need information on the structure of $\ker \rho$.

Notice that, by (8), for all elements $x, y, z, t \in L$ we have

$$j(x, y, z, t) := \{x, z, t\}y - \{y, z, t\}x - h(x, y, z, t) \in \ker \rho. \quad (11)$$

Let $J = J(L) = \text{vect}\{j(x, y, z, t) \mid x, y, z, t \in L\}$ be the vector space generated by all the elements of this kind; then evidently $J \subseteq S^2(L) \cap \ker \rho$. We will eventually show that $J = \ker \rho$; this will imply inclusion (10) and Theorem 1.

Our nearest objective is to prove that certain elements from $\ker \rho$ lie in J .

Let $\tilde{L} = LJ[X]$ and $A = A[X]$ be free Lie–Jordan and free associative algebras with free generators $X = \{x_1, x_2, \dots\}$. Denote by $\widetilde{SL} = SLJ[X]$ a *free special LJ-algebra*, that is, the LJ-subalgebra of A^\pm generated by X . Evidently, there exists an epimorphism $\varphi: \tilde{L} \rightarrow \widetilde{SL}$, identical on X . By definition of a universal enveloping algebra, φ extends uniquely to a homomorphism of associative algebras $\tilde{\varphi}: U(\tilde{L}) \rightarrow A$, $\tilde{\varphi}(x) = x$, $x \in X$. But A is a free algebra; hence $\tilde{\varphi}$ is an isomorphism. Thus, we have a homomorphism

$$\rho: \langle V(\tilde{L}), +, \cdot, [\ , \], \{ \ , \ , \} \rangle \rightarrow \text{gr } A = \widetilde{SL} \oplus A/\widetilde{SL}$$

such that $\rho(a) = \alpha(a)$, $a \in \tilde{L}$. Notice that the algebras $V(\tilde{L})$ and $\text{gr } A$ have natural gradings by degrees with respect to X , and the homomorphism ρ respects this grading.

The following proposition is crucial for our construction.

PROPOSITION 1. *In the above notation, every multilinear element of degree ≤ 6 on X from $\ker \rho$ lies in $J = J(\tilde{L})$.*

Proof. We will prove that J contains the multilinear elements of degree 6 from $\ker \rho$. For the elements of smaller degree the proof is similar but more easy.

Denote by P the subspace of multilinear elements of degree 6 in $V(\tilde{L})$. A subset W of a vector space V we will call a *pre-basis* of V if $\text{vect}\{W\} = V$. We have a decomposition $P = P_1 \oplus P_2$, where $P_1 = P \cap \tilde{L}$ and $P_2 = P \cap S^2(\tilde{L})$. Let us find a pre-basis of the space $(P_2 + J)/J$. Let $L[X]$ be a Lie subalgebra in \tilde{L} generated by X and let B be a homogeneous basis of the space of multilinear elements of $L[X]$. We have $B = \bigcup_{i=1}^{\infty} B_i$, where $B_i = \{b \in B \mid \deg_X b = i\}$. Set

$$\begin{aligned} Q_1 &= \{a_i b_j \mid a_i \in B_i, b_j \in B_j, i + j = 6\}, \\ Q_2 &= \{a_3 \{x, y, z\} \mid a_3 \in B_3, x, y, z \in X\}, \\ Q_3 &= \{a_2 \{b_2, y, z\} \mid a_2, b_2 \in B_2, y, z \in X\}, \\ Q_4 &= \{\{x, y, z\} \{u, v, w\} \mid x, y, z, u, v, w \in X\}. \end{aligned}$$

LEMMA 3. *The set $Q = \bigcup_{i=1}^4 Q_i$ is a pre-basis of $(P_2 + J)/J$.*

Proof. Let $v \in P_2$ be a monomial of degree 6. If $v \in L[X] \cdot L[X]$, then $v \in \text{vect}\{Q_1\}$.

If the operation $\{, , \}$ appears only once in v , then we can assume, by (3) and (4), that v has one of the forms

$$\begin{aligned} v_1 &= z\{a_3, x, y\}, & v_2 &= y\{a_2, b_2, x\}, \\ v_3 &= a_2\{b_2, x, y\}, & v_4 &= a_3\{x, y, z\}, \end{aligned}$$

where $a_i, b_i \in B_i, x, y, z \in X$. Now, for any $a, b, c, d \in \tilde{L}$ we have $h(a, b, c, d) \in \text{vect}\{Q_1\}$; hence, by (11),

$$\{a, c, d\}b - \{b, c, d\}a = j(a, b, c, d) + h(a, b, c, d) \in J + \text{vect}\{Q_1\}. \quad (12)$$

Therefore, by (3), $v_1, \dots, v_4 \in \text{vect}\{Q_1 \cup Q_2 \cup Q_3\} + J$.

Finally, if the operation $\{, , \}$ appears twice in v , then evidently

$$v = \sum_i \{\{x_i, y_i, z_i\}, t_i, u_i\} w_i \pmod{\text{vect}\{Q\}},$$

for some $x_i, \dots, w_i \in X$. By (12), we have

$$\{\{x_i, y_i, z_i\}, t_i, u_i\} w_i \equiv \{x_i, y_i, z_i\} \{w_i, t_i, u_i\} \pmod{\text{vect}\{Q\} + J},$$

so $v \in \text{vect}\{Q\} + J$, and Lemma 3 is proved. ■

Set $P_2^j = \text{vect}\{Q_1 + \dots + Q_j\} + J, j = 1, 2, 3, 4$, and $\bar{P}_2^j = P_2^j / P_2^{j-1}, j = 2, 3, 4$.

LEMMA 4. *We have*

- (i) $\dim(P_2^1/J) \leq 274$, (ii) $\dim \bar{P}_2^2 \leq 40$,
- (iii) $\dim \bar{P}_2^3 \leq 45$, (iv) $\dim \bar{P}_2^4 \leq 1$,
- (v) $\dim(P_2 + J/J) \leq 360$.

Proof. By [1, p. 50], the dimension of the space of multilinear elements of degree m in a free n -generated Lie algebra is equal to $B(n, m) = (m - 1)!C_m^n$. Therefore, we have

$$\begin{aligned} \dim P_2^1/J &\leq |Q_1| \leq B(6, 1)B(5, 5) + B(6, 2)B(4, 4) + B(6, 3)B(3, 3) \\ &= 144 + 90 + 40 = 274. \end{aligned}$$

Furthermore, by (2) and (3), we have for any $a_i \in L[X]$, $i = 1, 2, 3$, and $\sigma \in \text{Sym}\{1, 2, 3\}$

$$\{a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\} \equiv \{a_1, a_2, a_3\} \pmod{L[X]}, \quad (13)$$

which implies that $\dim \bar{P}_2^2 \leq B(6, 3) = 40$.

To prove (iii), notice that (12) implies

$$a_2\{b_2, x, y\} \equiv b_2\{a_2, x, y\} \pmod{L[X] + J};$$

hence $\dim \bar{P}_2^3 \leq 1/2B(6, 2)B(4, 2) = 45$.

For (iv), we need a certain corollary of identity (5). Adding (5) and the identity obtained from it by interchanging y and z , we get by (3)

$$\begin{aligned} &\{2\{x, y, z\} - [x, [y, z]], t, v\} \\ &= 2\{\{x, t, v\}, y, z\} - [\{x, t, v\}, [y, z]] - \{x, \{y, v, t\}, z\} \\ &\quad + \{x, y, \{z, t, v\}\} - \{x, \{z, v, t\}, y\} + \{x, z, \{y, t, v\}\} \\ &= 2\{\{x, t, v\}, y, z\} - [\{x, t, v\}, [y, z]] \\ &\quad - \{x, \{y, t, v\} + [y, [v, t]], z\} + \{x, y, \{z, v, t\} + [z, [t, v]]\} \\ &\quad - \{x, \{z, v, t\}, y\} + \{x, z, \{y, t, v\}\} \\ &= 2\{\{x, t, v\}, y, z\} - [\{x, t, v\}, [y, z]] - \{x, [y, [v, t]], z\} \\ &\quad + \{x, y, [z, [t, v]]\} - [x, [\{y, t, v\}, z]] + [x, [y, \{z, v, t\}]]. \end{aligned}$$

Since \tilde{L} is an LJ-algebra, this implies that for any $x_i \in X$, $i = 1, \dots, 6$, we have

$$\{\{x_1, x_2, x_3\}, x_4, x_5\} = \{\{x_1, x_4, x_5\}, x_2, x_3\} + S, \quad (14)$$

where S is a sum of multilinear elements from \tilde{L} on x_1, \dots, x_5 , in which the operation $\{, , \}$ appears only once. By the proof of Lemma 3, we have $S \cdot x_6 \in P_2^3$.

Now, for any $x_i \in X$, $i = 1, \dots, 6$, we have by (12)

$$\{\{x_1, x_2, x_3\}, x_4, x_5\}x_6 \equiv \{x_1, x_2, x_3\}\{x_4, x_5, x_6\} \pmod{P_2^3},$$

which implies by (14) that

$$\{x_1, x_2, x_3\}\{x_4, x_5, x_6\} \equiv \{x_1, x_4, x_5\}\{x_2, x_3, x_6\} \pmod{P_2^3}. \quad (15)$$

It follows easily from (13) and (15) that, for any $\sigma \in \text{Sym}\{1, \dots, 6\}$,

$$\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\}\{x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}\} \equiv \{x_1, x_2, x_3\}\{x_4, x_5, x_6\} \pmod{P_2^3}$$

holds, which implies (iv).

Finally, we have

$$\begin{aligned} \dim(P_2 + J/J) &\leq \dim P_2^1/J + \dim \bar{P}_2^2 + \dim \bar{P}_2^3 + \dim \bar{P}_2^4 \\ &= 274 + 40 + 45 + 1 = 360, \end{aligned}$$

proving the lemma. ■

LEMMA 5. $\dim P_1 \leq 360$.

Proof. Let us write $P_1 = P_1^0 + P_1^1 + P_1^2$, where P_1^i means the space generated by monomials from P_1 in which the operation $\{, , \}$ appears i times. Furthermore, for $i = 1, 2$ we set $\bar{P}_1^i = (P_1^0 + \dots + P_1^i)/(P_1^0 + \dots + P_1^{i-1})$. Clearly, $\dim P_1^0 \leq B(6, 6) = 5! = 120$.

Let B and B_i be as in Lemma 3. By (13), the union of the sets $\{B_4, X, X\}$, $\{B_3, B_2, X\}$, and $\{B_2, B_2, B_2\}$ forms a pre-basis of \bar{P}_1^1 . Taking (13) into account, we obtain

$$\dim \text{vect}\{\{B_4, X, X\}\} \leq B(6, 4) = 90,$$

$$\dim \text{vect}\{\{B_3, B_2, X\}\} \leq B(6, 3)B(3, 2) = 120,$$

$$\dim \text{vect}\{\{B_2, B_2, B_2\}\} \leq B(6, 2)B(4, 2)/3! = 15,$$

which implies that $\dim \bar{P}_1^1 \leq 90 + 120 + 15 = 225$.

Furthermore, it follows from (14) and (13) that the set $\{\{B_2, x_i, x_j\}, x_k, x_l\}$, where $x_s \in X$, $i < j < k < l$, forms a pre-basis of \bar{P}_1^2 ; hence $\dim \bar{P}_1^2 \leq B(6, 2) = 15$.

Thus, $\dim P_1 = \dim P_1^0 + \dim \bar{P}_1^1 + \dim \bar{P}_1^2 \leq 120 + 225 + 15 = 360$. ■

Return to the proof of Proposition 1. By Lemmas 4 and 5 we have that

$$\begin{aligned} \dim P/P \cap J &= \dim(P + J/J) \\ &\leq \dim P_1 + \dim(P_2 + J/J) \leq 360 + 360 = 720. \end{aligned}$$

On the other hand, the image $\rho(P)$ coincides with the space $(\text{gr } A)_{\langle 6 \rangle}$ of multilinear elements of degree 6 on X in $\text{gr } A$, which has the same dimension as the corresponding space $A_{\langle 6 \rangle}$ in A . Therefore, $\dim \rho(P) = \dim A_{\langle 6 \rangle} = 6! = 720$, and hence $P \cap \ker \rho = P \cap J$. ■

COROLLARY 1. $J = J(\tilde{L})$ is an ideal of the algebra $\langle V(L), +, \cdot, [\ , \], \{ \ , \ , \} \rangle$.

Proof. It suffices to prove that

$$[J, \tilde{L}] \subseteq J, \quad \{J, \tilde{L}, \tilde{L}\} \subseteq J.$$

Moreover, the space J is evidently stable with respect to endomorphisms of the space $V(\tilde{L})$ induced by those of the LJ-algebra \tilde{L} ; hence we need only to prove that for any $x, y, z, t, u, v \in X$ the inclusions

$$[j(x, y, z, t), u] \in J, \quad \{j(x, y, z, t), u, v\} \in J$$

hold. Observe that the elements under consideration belong to $\ker \rho$. Since they are multilinear and of degrees 5 and 6, they belong to J as well. ■

COROLLARY 2. The quotient algebra $\overline{V(\tilde{L})} = V(\tilde{L})/J(\tilde{L})$ is a Poisson LJ-algebra.

Proof. We have to prove that identities (1)–(5) hold in $V(\tilde{L})$ modulo $J(\tilde{L})$. If an identity contains at least two arguments from $S^2(\tilde{L})$, it holds trivially. Consequently, we can assume, as before, that all but one argument in the identities under consideration belong to X , and one argument belongs to $X \cup X \cdot X$. The identities evidently hold modulo $\ker \rho$. Since they are multilinear and of degree ≤ 6 on X , they hold modulo $J(\tilde{L})$ as well. ■

Notice that the constructions of $V(L)$ and $J(L)$ are functorial; that is, any LJ-algebra homomorphism $\varphi: L_1 \rightarrow L_2$ can be extended to a homomorphism

$$\tilde{\varphi}: \langle V(L_1), +, \cdot, [\ , \], \{ \ , \ , \} \rangle \rightarrow \langle V(L_2), +, \cdot, [\ , \], \{ \ , \ , \} \rangle,$$

and $\tilde{\varphi}(J(L_1)) = J(\varphi(L_1)) \subseteq J(L_2)$. Moreover, if φ is surjective then so is $\tilde{\varphi}$. Since every LJ-algebra can be represented as a quotient algebra of a free LJ-algebra, Corollaries 1 and 2 imply

PROPOSITION 2. For every LJ-algebra L the space $J(L)$ is an ideal of the algebra $V(L)$ with respect to the operations $\cdot, [\ , \], \{ \ , \ , \}$, and the quotient algebra $\overline{V(L)} = V(L)/J(L)$ is a Poisson LJ-algebra.

Since $J(L) \subseteq \ker \rho$, the homomorphism $\rho: V(L) \rightarrow \text{gr } U(L)$ induces a Poisson LJ-algebra homomorphism $\tilde{\rho}: \overline{V(L)} \rightarrow \text{gr } U(L)$, such that $\tilde{\rho}(l) =$

$\alpha(l)$ for any $l \in L$. In the next section we will prove that $\tilde{\rho}$ is an isomorphism, which will imply that $\ker \rho = J(L)$, proving inclusion (10) and Theorem 1.

4. $U(L)$ IS A DEFORMATION OF $\overline{V(L)}$

As we have seen, for the proof of Theorem 1 it suffices to prove that $\overline{V(L)}$ is isomorphic to $\text{gr } U(L)$. We will prove it by constructing an associative multiplication on $\overline{V(L)}$, isomorphic to that in $U(L)$.

Consider the space of polynomials $\overline{V(L)}[t]$ over $\overline{V(L)}$. We will call an associative multiplication $*$ on $\overline{V(L)}[t]$ a *quantization deformation* of $\overline{V(L)}$ if it satisfies the condition

- (i) $a * a = a^2$,
- (ii) $a * b - a * b = [a, b]t$,
- (iii) $a * b * c + c * b * a = \{a, b, c\}t^2$,
- (iv) $t * r = r * t$,

for any $a, b, c \in L$, $r \in \overline{V(L)}$.

The role of this definition is illustrated by the following.

PROPOSITION 3. *Let L be an LJ-algebra. If the Poisson LJ-algebra $\overline{V(L)}$ admits a quantization deformation, then L is special. Moreover, in this case $\text{gr } U(L)$ is isomorphic to $\overline{V(L)}$ and $U(L)$ is isomorphic to the quotient algebra $\langle A[t], +, * \rangle / A[t](1 - t)$.*

Proof. Assume that $\overline{V(L)} = L \oplus S^2(L)/J(L)$ admits a quantization deformation $*$. Define a new multiplication $a \star b = a * b|_{t=1}$ on $\overline{V(L)}$; then the algebra $B = \langle \overline{V(L)}, +, \star \rangle$ is associative, and, by properties (ii) and (iii), the LJ-algebra L is a subalgebra of the LJ-algebra B^\pm . Therefore, L is special.

Consequently, there exists an algebra homomorphism $\pi: U(L) \rightarrow B$ such that $\pi\alpha(l) = l$ for any $l \in L$. The algebra B is evidently generated by L and has a filtration $B_1 = L \subseteq B_2 = B$, where $B_2 = L + L \star L$; and it is easily seen that $\pi(U_k) \subseteq B_k$. Therefore, π induces a homomorphism of associated graded algebras $\tilde{\pi}: \text{gr } U \rightarrow \text{gr } B$, with $\tilde{\pi}\alpha(l) = l$ for any $l \in L$. But $\text{gr } B$ is easily seen to be isomorphic to $\overline{V(L)}$, and we have seen that there exists a graded algebra homomorphism $\tilde{\rho}: \overline{V(L)} \rightarrow \text{gr } U(L)$, such that $\tilde{\rho}(l) = \alpha(l)$ for any $l \in L$. This proves that $\text{gr } U(L)$ is isomorphic to $\overline{V(L)}$. Observe that $U(L)$ and $\text{gr } U(L)$ are isomorphic as vector spaces. Choose in $U(L)$ a base $\{u_i\}$ such that the set $\{\bar{u}_i\}$ is a graded base of $\text{gr } U(L)$; then evidently $\{\pi(u_i)\}$ is a base of the space $\overline{V(L)}$, and so $\ker \pi = 0$ and π is an isomorphism. ■

Now, everything is finished by the following.

THEOREM 2. *For any LJ-algebra L , the Poisson LJ-algebra $\overline{V(L)}$ admits a quantization deformation.*

Proof. We first define a multiplication $*$ on $V(L)[t]$ by setting

$$\begin{aligned} a * b &= ab + 1/2[a, b]t, \\ a^2 * b &= 1/2([a^2, b]t + \{a, a, b\}t^2), \\ b * a^2 &= 1/2([b, a^2]t + \{a, a, b\}t^2), \\ a^2 * b^2 &= 1/2(\{a, a, b^2\}t^2 + [\{a, a, b\}, b]t^3), \end{aligned}$$

Let us check that $J(L) * V(L) + V(L) * J(L) \subseteq J(L)[t]$. Let $x, y, z, r \in L$; consider

$$\begin{aligned} j(x, y, z, r) &= \{x, z, r\}y - \{y, z, r\}x - 1/2[z, [x, y]]r \\ &\quad - 1/2([r, [x, y]]z + [[r, z], x]y - [[r, z], y]x) \\ &= \sum_i j_1^{(i)} j_2^{(i)}. \end{aligned}$$

We claim that for any $a \in V(L)$

$$\sum_i \{j_1^{(i)}, j_2^{(i)}, a\} + \{j_2^{(i)}, j_1^{(i)}, a\} \in J \quad (16)$$

holds.

In fact, we have

$$\begin{aligned} &\sum_i \{j_1^{(i)}, j_2^{(i)}, a\} + \{j_2^{(i)}, j_1^{(i)}, a\} \\ &= \{\{x, z, r\}, y, a\} + \{y, \{x, z, r\}, a\} - \{\{y, z, r\}, x, a\} \\ &\quad - \{x, \{y, z, r\}, a\} - 1/2\{[z, [x, y]], r, a\} - 1/2\{r, [z, [x, y]], a\} \\ &\quad - 1/2\{[r, [x, y]], z, a\} - 1/2\{z, [r, [x, y]], a\} \\ &\quad - 1/2\{[[r, z], x], y, a\} - 1/2\{y, [[r, z], x], a\} \\ &\quad + 1/2\{[[r, z], y], x, a\} + 1/2\{x, [[r, z], y], a\}. \end{aligned}$$

One can straightforwardly check that this sum is identically zero in the algebra \widetilde{SL} . Since it is of degree 5, by Proposition 1 it is an identity in the free LJ-algebra \tilde{L} . Consequently, this identity holds in the LJ-algebra $\overline{V(L)} = V(L)/J(L)$, which implies inclusion (16).

Now, by definition of the multiplication $*$, we have

$$j(x, y, z, r) * a = 1/2[j, a]t + 1/4 \sum_i (\{j_1^{(i)}, j_2^{(i)}, a\} + \{j_2^{(i)}, j_1^{(i)}, a\})t^2,$$

which lies in $J[t]$ by Corollary 1 and (16). Furthermore, if $a \in S^2(L)$, $a = b^2$, then

$$\begin{aligned} j(x, y, z, r) * b^2 &= 1/2 \sum_i (\{j_1^{(i)}, j_2^{(i)}, b^2\} + \{j_2^{(i)}, j_1^{(i)}, b^2\})t^2 \\ &\quad + 1/2 \sum_i [\{j_1^{(i)}, j_2^{(i)}, b\} + \{j_2^{(i)}, j_1^{(i)}, b\}, b]t^3, \end{aligned}$$

which lies in $J[t]$ by the same reasons. Therefore, $V(L) * J(L) \subseteq J(L)[t]$. Similarly, $J(L) * V(L) \subseteq J(L)[t]$.

The inclusions just proved show that $*$ can be defined naturally on the quotient algebra $\bar{V}(L)[t]$. Conditions (i)–(iv) are evidently satisfied by $*$. We have only to prove that $*$ is associative.

We will consider only the most complicated case; the remaining cases are proved similarly.

Compute

$$(a^2 * b^2) * c^2 = 1/2(\{a, a, b^2\} * c^2)t^2 + 1/2([\{a, a, b\}, b] * c^2)t^3.$$

One can easily check that in any Poisson LJ-algebra the identities

$$\{a, a, b^2\} = \{a, a, b\}b + [a, b]^2 + a[[a, b], b].$$

hold. Hence,

$$\begin{aligned} \{a, a, b^2\} * c^2 &= 1/4(\{\{a, a, b\}, b, c^2\} + \{b, \{a, a, b\}, c^2\})t^2 \\ &\quad + 1/4(\{a, [[a, b], b], c^2\} + \{[[a, b], b], a, c^2\})t^2 \\ &\quad + 1/2[\{a, b\}, [a, b], c^2]t^2 + 1/2[\{[a, b], [a, b], c\}, c]t^3 \\ &\quad + 1/4(\{\{a, a, b\}, b, c\}, c + \{b, \{a, a, b\}, c\}, c)t^3 \\ &\quad + 1/4(\{a, [[a, b], b], c\}, c + \{[[a, b], b], a, c\}, c)t^3. \end{aligned}$$

Therefore, $(a^2 * b^2) * c^2 = At^4 + Bt^5$, where

$$\begin{aligned} A &= 1/8(\{\{a, a, b\}, b, c^2\} + \{b, \{a, a, b\}, c^2\} + \{a, [[a, b], b], c^2\} \\ &\quad + \{[[a, b], b], a, c^2\} + 2[[a, a, b], b], c^2) \\ &\quad + 2[\{a, b\}, [a, b], c^2) \end{aligned}$$

and

$$\begin{aligned} B = 1/8 & ([\{\{a, a, b\}, b, c\}, c] + [\{b, \{a, a, b\}, c\}, c] \\ & + [\{a, [[a, b], b], c\}, c] + [\{[[a, b], b], a, c\}, c] \\ & + 2[\{[a, b], [a, b], c\}, c] + 2\{c, c, [\{a, a, b\}, b\})). \end{aligned}$$

Similarly, $a^2 * (b^2 * c^2) = A_1 t^4 + B_1 t^5$, where

$$A_1 = 1/4(\{a, a, \{b, b, c^2\}\} + [a^2, [\{b, b, c\}, c]])$$

and

$$\begin{aligned} B_1 = 1/8 & ([\{a, a, b\}, \{b, b, c\}] + [\{a, a, \{b, b, c\}\}, c] \\ & + [\{a, a, b\}, [[b, c], c]] + [\{a, a, [[b, c], c]\}, b] \\ & + 2[\{a, a, [b, c]\}, [b, c]] + 2\{a, a, [\{b, b, c\}, c]\}). \end{aligned}$$

One can straightforwardly check that $A = A_1$ and $B = B_1$ in the algebra $\text{gr } A[X]$ for any $a, b, c \in X$. Since the elements $A - A_1$, $B - B_1$ are of degree 6, by Proposition 1 we have $A - A_1$, $B - B_1 \in J(L)$, and hence $A = A_1$, $B = B_1$ in $\overline{V(L)}$.

This proves the theorem. ■

5. APPLICATION

To formulate an application of our result, we recall some definitions from [2].

Let $A = A[x, y]$ be the free associative algebra over a field F of characteristic $p > 0$ on free generators x, y . For any $f = f(x, y) \in A$ we define a space A_f as a minimal subspace of A with the properties

- (i) $x, y \in A_f$,
- (ii) if $a, b \in A_f$, then $f(a, b) \in A_f$.

A series $E(x) = \sum_{i=0}^{\infty} a_i x^i$ is called an f -exponent if $a_0 = 1$ and in the natural completion \hat{A} of A we have

$$E(x)E(y) = E(z), \quad (17)$$

where $z = \sum_{i=0}^{\infty} z_i$, $\deg z_i = i$, $z_i \in A_f$.

A series $E(x)$ we will call a p -exponent if $E(x)$ is an f -exponent for some polynomial f , and for any other polynomial g we have $A_f \subseteq A_g$.

It was proved in [2] that in the case $p = 0$ a 0-exponent exists and coincides with the classical exponent $\exp x = \sum_{i=0}^{\infty} x^i / i!$. It was also con-

jectured there that a p -exponent exists (certainly not unique!) and coincides with an f_p -exponent, where the element $f_p \in A$ is defined as follows.

It is well known (see for instance [3, Sect. 5.7]) that for every simple $p > 0$ there holds an equality of the type

$$(x + y)^p = x^p + y^p + s(x, y) + pf_p(x, y),$$

where $s(x, y)$ and $\tilde{f}_p(x, y)$ are a Lie and an associative polynomial on x, y with integer coefficients. Of course, the polynomial $\tilde{f}_p(x, y)$ is not uniquely defined by this equation, but it is unique modulo Lie polynomials.

Now, we fix \tilde{f}_p , reduce its coefficients modulo p , and set $f_p(x, y) = [x, y] + \tilde{f}_p(x, y)$.

For $p = 3$, it is easy to see that we can take $\tilde{f}_3 = xyx + yxy$; hence $f_3 = xy - yx + xyx + yxy$, and the subspace A_{f_3} coincides with the free special LJ-algebra $SLJ[x, y]$. Consequently, the elements z_i in equality (17) belong to $SLJ[x, y]$ and can be written as concrete polynomials on x, y in terms of operations $[,]$ and $\{ , , \}$. More specifically, it was proved in [2] that the series

$$E(x) = x + \sqrt{1 + x^2} = 1 + x - x^2 + x^4 + \dots$$

is a 3-exponent and an f_3 -exponent; and the element z from (17) is given by

$$\begin{aligned} z &= E^{-1}(E(x)E(y)) = x * y \\ &= x + y - [x, y] + \{x, x, y\} + \{x, y, y\} + \dots \end{aligned} \quad (18)$$

The following theorem was conjectured in [2].

THEOREM 3. *Let $L = LJ[x, y]$ be the free LJ-algebra over a field F of characteristic three on generators x, y , and let \bar{L} be the completion of L ; then the operation $u * v = w$, defined by (18), introduces on \bar{L} a group structure.*

Proof. By Theorem 1, the LJ-algebra L is special; hence it is isomorphic to $SL = SLJ[x, y]$, and by (17) we have $u * v = E^{-1}(E(u)E(v))$ for any $u, v \in L = SL \subseteq A$. This easily implies the statement of the theorem.

■

REFERENCES

1. Yu. A. Bahturin, "Identities of Lie Algebras," Nauka, Moscow, 1985. (In Russian; English translation by VNU Science Press, Utrecht, 1987.)
2. A. Grishkov, An analog of exponential map and Campbell-Hausdorff series in characteristic 3, in preparation.

3. N. Jacobson, "Lie Algebras," Interscience, New York, 1962.
4. N. Jacobson, "Structure and Representations of Jordan Algebras," Am. Math. Soc., Providence, 1968.
5. I. P. Shestakov, Every Akivis algebra is linear, *Geom. Dedicata* **77** (1999), 215–223.
6. I. P. Shestakov, Specialty and deformations of algebras, in "Proceedings of the International Algebraic Conference Dedicated to Memory of A. G. Kurosh," May 1998, Moscow, pp. 345–356, de Gruyter, Berlin, 2000.
7. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, "Rings That Are Nearly Associative," Nauka, Moscow, 1978. (In Russian; English translation by Academic Press, New York, 1982.)